MS221 CB A

The Open University

A second level interdisciplinary course

Exploring Mathematics

COMPUTER BOOK

A

BLOCK A

MATHEMATICAL EXPLORATION

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Prepared by the course team

About this course

This computer book forms part of the course MS221 *Exploring Mathematics*. This course and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software package Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1997. Second edition 2001. New edition 2004.

Edited, designed and typeset by The Open University, using the Open University TEX System.

Printed and bound in the United Kingdom by The Charlesworth Group, Huddersfield.

ISBN 0 7492 6651 1

3.1

Contents

Guidance notes

This computer book contains those sections of the chapters in Block A which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter A1

221A1-01 Fibonacci numbers
221A1-02 Linear second-order recurrence sequences
221A1-03 A Fibonacci sunflower (Optional)

Chapter A2

221A2-01 Ellipses in parametric form
221A2-02 Hyperbolas in parametric form
221A2-03 Focus, directrix and eccentricity (Optional)

Chapter A3

221A3-01 Isometries and triangles
221A3-02 Isometries and conics
221A3-03 Surface and contour plots (Optional)

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for each chapter also require you to work with Mathcad documents which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter A1, Section 4
Exploring linear second-order recurrence sequences with the computer

Mathcad can quickly produce tables of terms of linear second-order recurrence sequences, and of expressions related to such sequences. These can help you to spot patterns in the sequences, and so form conjectures about them. In this section you will be introduced to the Mathcad techniques needed to produce such tables. You will then use examples of these tables to look for patterns and to form conjectures.

The section ends with an *optional* subsection in which you are invited to explore the relationship between the pattern of florets on a sunflower head and the Fibonacci numbers, using a Mathcad file that plots a sunflower pattern.

Further information on Mathcad features can be found in the MS221 reference manual *A Guide to Mathcad*.

4.1 Computer exploration of Fibonacci numbers

In the first activity, you will see how to use Mathcad to calculate terms of the Fibonacci sequence F_n using the recurrence system

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 0, 1, 2, \ldots), \tag{4.1}$$

See Chapter A1, Section 2.

and how to display these terms in a convenient table.

Activity 4.1 The Fibonacci recurrence system

Open Mathcad file **221A1-01 Fibonacci numbers**. Page 2 of the worksheet describes some key features of MS221 Mathcad worksheets.

The Mathcad worksheets for MS221 have a similar design to those for MST121. Remember to make your own working copies of them.

When you have finished with page 2, move to page 3 of the worksheet and carry out Task 1.

A solution is given on page 4 of the worksheet.

Comment

- Range variables and subscripted variables play essential roles in calculating terms of a recurrence sequence in Mathcad. At the beginning of Task 1, you defined the range variable n as follows:

  $$n := 0, 1 \,..\, N - 2.$$

  This ensures that the subsequent definitions of the subscripted variables F_0, F_1 and F_{n+2},

  $$F_0 := 0, \quad F_1 := 1, \quad F_{n+2} := F_{n+1} + F_n,$$

  together define *all* the Fibonacci numbers $F_0, F_1, \ldots, F_N$, *and no others*. Mathcad carries out the final definition here $N - 1$ times, once for each of the values $0, 1, \ldots, N - 2$ in the range of n. As it does so, the subscript $n + 2$ takes each of the values in the range $2, 3, \ldots, N$ in turn.

n	0	1	$\cdots$	$N-2$
$n+2$	2	3	$\cdots$	N

- You displayed the Fibonacci numbers $F_0, F_1, \ldots, F_N$ by entering '$F =$'. This causes all the values of the subscripted variable to appear in a table. The table displays the subscripts on the left (if there are more than nine values), and scrolls if there are many values to display.
- You defined the variable N to be 20, to calculate the Fibonacci numbers $F_0, F_1, \ldots, F_{20}$. You could calculate more Fibonacci numbers, by increasing the value of N. Mathcad automatically updates the values of all other variables whose definitions appear subsequently and depend on N, for example, the range variable $n := 0, 1 \,..\, N - 2$.

If you increase the value of N, remember to set it to 20 again before continuing.

In Activity 4.1, you saw how to calculate and display terms of a sequence defined by a recurrence system. In the next activity, you will see a simple way to calculate and display terms of a sequence defined by a *formula* involving a variable such as n. This method will be used to produce tables showing the sequences defined by Binet's formula,

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \quad (n = 0, 1, 2, \ldots),$$

and by Binet's approximation, $\phi^n/\sqrt{5}$.

See Chapter A1, Section 3. Recall that

$$\phi = \tfrac{1}{2}(1 + \sqrt{5}),$$
$$\psi = \tfrac{1}{2}(1 - \sqrt{5}).$$

Activity 4.2 Binet's formula and Binet's approximation

You should still be working with Mathcad file 221A1-01.

Move to page 5 of the Mathcad worksheet, and carry out Task 2. Compare the values calculated using Binet's formula and Binet's approximation with the Fibonacci numbers $F_0, F_1, F_2, \ldots, F_{20}$, which were calculated using the recurrence system (4.1).

Comment

- The table for Binet's formula shows that the formula does indeed give the same value for F_n as the recurrence system, for $n = 0, 1, \ldots, 20$.

  The table for Binet's approximation makes it easy to see that F_n is indeed the nearest integer to the value given by Binet's approximation, for $n = 0, 1, \ldots, 20$. You can also see that, as n increases through the range $0, 1, \ldots, 20$, the values of Binet's approximation alternate above and below the Fibonacci numbers, and become progressively closer to them.
- The tables of values on page 5 of the Mathcad worksheet are produced by evaluating formulas involving a range variable which has been defined previously. Each table displays the values taken by the corresponding formula as the range variable takes each value in its range. By default, the table scrolls if there are many values to display, but it can be resized to see all of the values without scrolling.

  Here, the range variable is n; its definition, $n := 0, 1 \,..\, N$, appears after the words 'Display range'. The first table is for the formula n and therefore displays $0, 1, \ldots, N$. The second is for F_n, so this table displays $F_0, F_1, \ldots, F_N$, which were all defined on page 3 of the worksheet. The other two tables are for Binet's formula and Binet's approximation.

Mathcad notes

Mathcad notes provide extra information about the features and techniques used in the Mathcad files. They are *optional*.

- Mathcad variables can be defined more than once, so they can take different values at different places in a worksheet. For example, the range variable n is defined as $n := 0, 1 .. N - 2$ on page 3 (to calculate the Fibonacci numbers) and as $n := 0, 1 .. N$ on page 5 (to display them).

  Each definition of a variable is used by Mathcad for all expressions involving the variable which appear below that definition in the worksheet, up to the point where a new definition of the same variable appears or the worksheet ends.

  On the same line and to the right counts as 'below', but on the same line and to the left counts as 'above'.
- The table of Fibonacci numbers on page 5 of the worksheet appears different in style from that on page 4. The table obtained by entering '$F_n =$' on page 5 is displayed below the expression, whereas that obtained by entering '$F =$' on page 4 is displayed to the right, with an additional grey column to denote the subscript values. These differences (between tables containing the same output values) are due to different Mathcad display defaults being triggered in either case. The table characteristics can be altered by clicking on a value in the table with the *right* mouse button, and then choosing either 'Properties...' or 'Alignment' from the resulting mini-menu.
- To enter a Greek letter in Mathcad, you can either click on the appropriate button on the 'Greek' toolbar, or type the equivalent Roman letter and then press `[Ctrl]g`. For example, if you type **a** followed by `[Ctrl]g`, then the 'a' will change into an 'α' (alpha).

Now close Mathcad file 221A1-01.

4.2 Exploring patterns in linear second-order recurrence sequences

In the next series of activities, you are invited to explore the patterns that occur when the terms of linear second-order recurrence sequences are combined in certain ways. First you will be asked to consider two different formulas involving the terms of the Fibonacci sequence, namely

$$\frac{F_{n+1}}{F_n} \quad \text{and} \quad F_n^2 + F_{n+1}^2,$$

and to look for patterns in the sequences defined by these formulas. You will then be asked to explore how these patterns generalise to other linear second-order recurrence sequences.

Mathcad file 221A1-02 has been set up to help you to carry out this investigation – the worksheet displays the sequences defined by the above formulas. However, the notation u_n is used rather than F_n in the worksheet, because you can use the same worksheet to see the sequences corresponding to *any* linear second-order recurrence sequence u_n (subject to the accuracy of Mathcad calculations). You can do this just by changing the values of the initial terms a and b, and the coefficients p and q, in the recurrence system

$$u_0 = a, u_1 = b, \quad u_{n+2} = pu_{n+1} + qu_n \quad (n = 0, 1, 2, \ldots),$$

which defines the sequence u_n in the file.

Activity 4.3 Patterns in the Fibonacci sequence

Open Mathcad file **221A1-02 Linear second-order recurrence sequences.** The worksheet is set up with $p = 1$, $q = 1$, $a = 0$ and $b = 1$, so u_n is the Fibonacci sequence. It displays the sequences

Mathcad displays u_n^2 as $(u_n)^2$.

$$n, \quad u_n, \quad \frac{u_{n+1}}{u_n} \quad \text{and} \quad u_n^2 + u_{n+1}^2.$$

Look at each of the last two sequences in turn. For each sequence, try to spot a pattern, and hence make a conjecture about a general property of the Fibonacci sequence.

The solutions use the notation F_n, since the sequence here *is* the Fibonacci sequence, but you may choose to use u_n instead.

Solutions are given on page 33.

Mathcad notes

- ◇ You may wonder why the range variable n used to display the tables is defined only to take values in the range $1, 2, \ldots, N - 1$, when the recurrence system defines all the terms of the sequence from u_0 to u_N. Notice, however, that if we increase the final value of the range variable n from $N - 1$ to N, then the last term in the 'ratio' table should be u_{N+1}/u_N. This term cannot be calculated, since the term u_{N+1} was not defined earlier in the file. Also, if the first value of the range variable is 0, then the first term in the 'ratio' table should be u_1/u_0, which cannot be calculated, since $u_0 = 0$.

If you try to display a table which contains a value that cannot be calculated, then Mathcad registers an error and shows no values at all.

- ◇ The default result format has been set so that 'Number of decimal places' is 9. Also, 'Exponential threshold' has been set to 15, so numbers between 10^{-15} and 10^{15} are shown in ordinary decimal notation. (Note that these settings affect only how Mathcad *displays* numbers – it always stores values internally to 15 significant figures for calculation purposes.)
- ◇ All the tables have been resized, to show 19 values without scrolling (for the range $1, 2, \ldots, 20 - 1$). If N is increased so that there are more values to display, then each table will scroll.

In Activity 4.3, you saw that the sequence F_{n+1}/F_n of ratios of successive terms of the Fibonacci sequence appears to tend to the golden ratio ϕ, and to alternate above and below ϕ. The golden ratio ϕ is one of the roots of the auxiliary equation associated with the Fibonacci sequence.

See Chapter A1, Section 3.

In the next activity you are asked to use the same worksheet to explore the sequence u_{n+1}/u_n of ratios for a number of different linear second-order recurrence sequences u_n. You can change the recurrence sequence in the worksheet by altering the values of p, q, a and b.

You will probably find it helpful to scroll down the worksheet until the line containing the definitions of p, q, a and b is at the top of your screen, before you begin Activity 4.4. This should allow you to see enough values of the sequence to spot any patterns, without having to scroll up and down every time you re-define p, q, a and b.

Activity 4.4 Exploring u_{n+1}/u_n

For each set of values of p, q, a and b suggested in Table 4.1, look at the sequence u_{n+1}/u_n and make a note of any pattern you observe. Note that the file displays the roots of the auxiliary equation associated with the sequence u_n – which should be helpful! There are some spaces in the table for you to choose your own values of a and b, and you may wish to extend the table on a separate sheet to try further values of p and q.

You should still be working with Mathcad file 221A1-02. Ignore the sequence $u_n^2 + u_{n+1}^2$ for the moment.

Note that the table of ratios may register a 'Found a singularity ...' error for some combinations of p, q, a and b. This indicates an attempt to divide by zero, which happens here if one of the terms $u_1, u_2, \ldots, u_{N-1}$ of the original sequence is zero.

If an error occurs, then Mathcad highlights the offending expression in red. Clicking on this expression reveals an error message.

When you have completed the table, try to make a conjecture about the long-term behaviour of the sequence u_{n+1}/u_n.

Table 4.1

Coefficients p	q	Initial values a	b	Apparent pattern in u_{n+1}/u_n $(n = 1, 2, \ldots)$
1	1	0	1	tends to ϕ, alternates above and below ϕ
1	1	2	1	tends to ϕ
1	1			
−1	1	0	1	tends to $-\phi$
−1	1	−1	1	tends to $-\phi$
−1	1			
3	−1	2	1	tends to 2.618
3	−1			
3	−1			

Solutions are given on page 33.

In Activity 4.3 you met the conjecture that the sequence $F_n^2 + F_{n+1}^2$ consists of every other term of the Fibonacci sequence. In the next activity you are asked to explore the sequence $u_n^2 + u_{n+1}^2$ for a number of different linear second-order recurrence sequences u_n.

Activity 4.5 Exploring $u_n^2 + u_{n+1}^2$

For each set of values of p, q, a and b suggested in Table 4.2 overleaf, look at the sequence $u_n^2 + u_{n+1}^2$, and make a note of any pattern you observe.

You should still be working with Mathcad file 221A1-02.

When you have completed the table, try to make a conjecture about the sequence $u_n^2 + u_{n+1}^2$ which is more general than the above conjecture about $F_n^2 + F_{n+1}^2$, in cases where $p = 1$, $q = 1$ and either $a = 0$ or $b = 0$.

If you have time, you may also like to look at what happens if you vary p.

Table 4.2

Coefficients		Initial values		Apparent pattern in
p	q	a	b	$u_n^2 + u_{n+1}^2$, $n = 1, 2, \ldots$
1	1	0	1	it is every other term of u_n, starting with u_3
1	1	0	2	
1	1	0	3	
1	1	1	0	
1	1	2	0	
1	1	3	0	

Solutions are given on page 33.

Now close Mathcad file 221A1-02.

4.3 Fibonacci numbers in sunflowers (Optional)

Do not be tempted to spend too long on this subsection!

The Mathcad file associated with this subsection allows you to investigate the close association between the pattern of florets on a sunflower head and the Fibonacci numbers.

See Chapter A1, Subsection 2.2.

You may remember that the florets of a sunflower develop from tiny lumps called primordia, which are created one after another on the edge of a disc in the centre of the sunflower head and then move outwards. In most sunflowers, the angle between successive primordia is about 137.5°. This is very close to $360(1 - 1/\phi)°$, where ϕ is the golden ratio.

The Mathcad file simulates the process that takes place on a sunflower head, and the pattern that it produces is shown in Figures 4.1 and 4.2. The file allows you to see what happens if you choose angles other than 137.5°. It also allows you to investigate the spirals that appear in the sunflower pattern. Some clockwise spirals are highlighted in Figure 4.1, and some anticlockwise spirals are highlighted in Figure 4.2. We say here that a spiral is clockwise if your finger moves clockwise when tracing it inwards towards the centre, and anticlockwise otherwise.

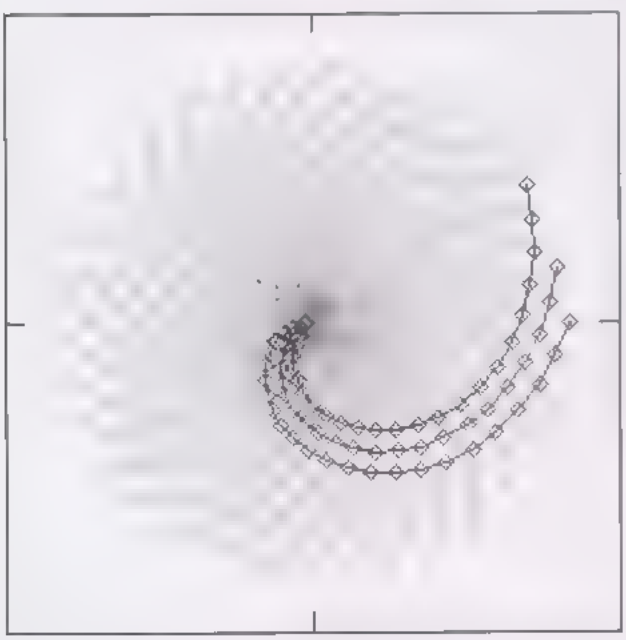

Figure 4.1 Clockwise spirals

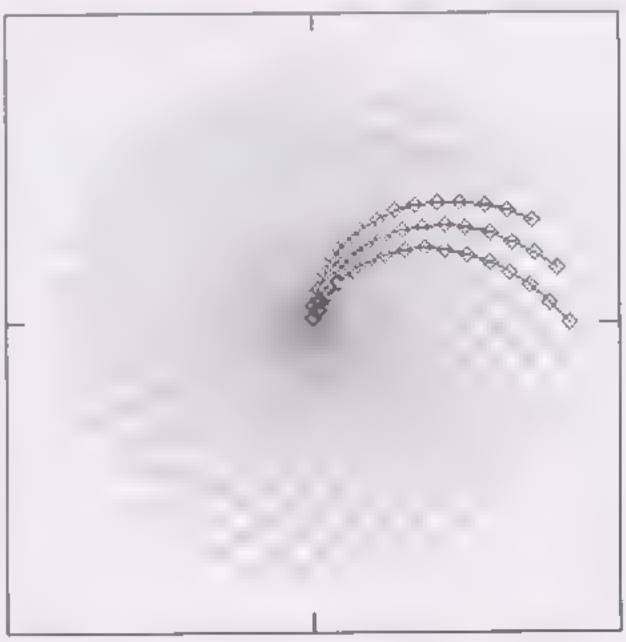

Figure 4.2 Anticlockwise spirals

Activity 4.6 A Fibonacci sunflower (Optional)

Open Mathcad file **221A1-03 A Fibonacci sunflower**, and experiment in the ways described in the worksheet.

Comment

It seems that most choices of the angle between successive points produce a less well-packed pattern than that produced by the angle 137.5°.

In the second part of the investigation, the angle between successive points is left fixed as 137.5°. It appears that highlighting every fth point produces a visible spiral if f is a Fibonacci number (other than 1, 2 or 3), but that no such spiral is produced for most other numbers f.

It seems also that as f increases through the Fibonacci sequence, the spirals produced are alternately clockwise and anticlockwise, and progressively less 'curly'.

Now close Mathcad file 221A1-03.

It is indeed true that Fibonacci numbers always give rise to spirals in a 137.5° sunflower pattern in the way that we have described, and that these spirals have the properties mentioned in the comment above. If you are interested in the reasons behind these facts, you may like to read the explanation given at the end of this subsection.

For each Fibonacci number f with $f \geq 5$, the spiral shown in the Mathcad file is just one of a set of spirals which do not intersect each other but together contain all the points in the sunflower pattern. The number of spirals in the set is f. In any particular sunflower pattern, just some of these sets of spirals are obvious to the eye - the ones in which successive points in each spiral are close together.

Figures 4.1 and 4.2 show some of the spirals corresponding to $f = 21$ and $f = 34$.

You may be wondering how commonly real sunflowers display the type of pattern that we have explored. Research has established that approximately 95% of sunflowers are 'Fibonacci', where the angle between successive primordia is about 137.5°. Approximately 5% are 'Lucas', where the angle between successive primordia is about 99.5°, and the numbers of spirals are Lucas numbers. A small proportion of sunflowers do not fall into either of these two categories. Research also shows that successive primordia may appear in clockwise or anticlockwise order: 50% of sunflowers are clockwise and 50% anticlockwise.

This botanical information was kindly provided by Prof. Ralph O. Erickson from the University of Pennsylvania.

The Lucas sequence is $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$. See Chapter A1, Exercise 3.2.

Explanation of the spirals observations

The key to proving that the observations in the Comment following Activity 4.6 are true in general turns out to be the identity

$$F_{n+1} - \phi F_n = \psi^n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.2}$$

which can be derived from Binet's formula, as you will see in Section 5.

The sunflower pattern in the Mathcad file is produced by plotting 500 points. The first point is the one on the extreme right-hand side, and each subsequent point is plotted at an angle of approximately $360(1 - 1/\phi)$ clockwise of the previous point, and a small distance nearer the centre.

All angles in this discussion are measured in degrees.

Since

$$360\left(1-\frac{1}{\phi}\right)=360-\frac{360}{\phi},$$

the *anticlockwise* angle from each point to the next point in the sunflower pattern is $360/\phi \simeq 222.5$. Suppose that we choose the Fibonacci number F_n, and highlight every F_nth point. Then the anticlockwise angle from each highlighted point to the next highlighted point is

$$F_n \times \frac{360}{\phi}.$$

By equation (4.2) (with n replaced by $n-1$), this angle is equal to

$$(\phi F_{n-1} + \psi^{n-1}) \times \frac{360}{\phi} = (F_{n-1} \times 360) + \left(\frac{\psi^{n-1}}{\phi} \times 360\right).$$

n	F_n	$\frac{\psi^{n-1}}{\phi} \times 360$
2	1	−137.508
3	2	84.984
4	3	−52.523
5	5	32.461
6	8	−20.062
7	13	12.399
8	21	−7.663
9	34	4.736
10	55	−2.927

The first term in the expression on the right, $F_{n-1} \times 360$, corresponds to F_{n-1} complete turns, and so the anticlockwise angle from each highlighted point to the next is just

$$\frac{\psi^{n-1}}{\phi} \times 360.$$

Now $\psi = -0.618\ldots$, so ψ^{n-1} alternates in sign and tends to 0 as n tends to infinity. It follows that the above sequence of angles behaves in the same way. Its terms for $n = 2, 3, \ldots, 10$ are shown to three decimal places in the table in the margin.

Remember that the *first* highlighted point is the one furthest from the centre of the sunflower pattern.

Thus, if n is even and large enough, then each highlighted point after the first is the same *small* number of degrees *clockwise* of the preceding highlighted point. Since it is also slightly closer to the centre, a clockwise spiral is produced. This is illustrated in Figure 4.1, which includes the spiral starting from the first point in the pattern when $n = 8$. There is a similar spiral starting from each of the first F_8 points, making a set of $F_8 = 21$ clockwise spirals in all, and three of these are shown in the figure. Similarly, if n is odd and large enough, then each highlighted point after the first is the same *small* number of degrees *anticlockwise* of the preceding highlighted point, and an anticlockwise spiral is produced. Figure 4.2, where $n = 9$, shows three of the set of $F_9 = 34$ anticlockwise spirals.

So for all large enough Fibonacci numbers F_n there are F_n spirals (provided that there are enough points in the sunflower pattern!), and the spirals are clockwise or anticlockwise according as n is even or odd. Also, since the magnitude of the angle between successive highlighted points decreases as n increases, bigger Fibonacci numbers produce less 'curly' spirals.

Chapter A2, Section 6
Conics on the computer

In this section you will learn how to plot conics on the computer, using parametric representations.

At the end of the section, there is an *optional* subsection in which you are invited to use Mathcad to explore how the shape of a conic changes as its eccentricity changes.

6.1 Ellipses

The standard parametrisation of the ellipse is

$$x = a\cos t, \quad y = b\sin t \quad (0 \leq t \leq 2\pi).$$

See Chapter A2, Subsection 5.1.

This is a parametrisation of the ellipse in standard position with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a \geq b > 0).$$

In the first activity you will plot ellipses in standard position, and a translated ellipse.

Activity 6.1 Plotting ellipses from parametric equations

Open Mathcad file **221A2-01 Ellipses in parametric form**. Page 2 of the worksheet describes some basic Mathcad techniques for producing graphs.

These techniques are used in the computer work for Chapter A2 of MST121, so you may be familiar with them already.

When you have finished with page 2, work through pages 3 and 4 of the worksheet, carrying out Tasks 1 and 2.

Comments are provided in the worksheet, and there is a further comment below.

Comment

When you enter expressions involving a range variable in the x- and y-axis placeholders of a graph, Mathcad plots one point for each value of the range. By default, Mathcad joins these points together with line segments, to produce a continuous curve. The result is an approximation to the true curve – the more points plotted, the more accurate the approximation.

As a rule of thumb, the range variable used to plot a curve should have a step size which ensures that at least 100 points are plotted. For the ellipses, the graph range $t := 0, 0.01 .. 2\pi$ (a step size of 0.01) was used, which gives over 600 points. Note that the smaller you make the step size, the longer Mathcad will take to plot the graph, and once you reduce the step size beyond a certain level there will be no noticeable improvement in the shape of the curve. It is usually not worth plotting more than 1000 points.

Remember that Mathcad notes are *optional*.

Mathcad notes

The technique used to plot two curves on a graph (such as the two ellipses on page 4 of the worksheet) can be extended to plot three, four, or more curves on the same graph. All the expressions for x should be entered in the x-axis placeholders, separated by commas, and the expresssions for y should be entered likewise in the y-axis placeholders. Mathcad plots the first y-axis expression against the first x-axis expression, the second against the second, and so on. When the graph is plotted, the line style and colour used to display each curve appear underneath the expressions on the y-axis.

When a curve is represented by parametric equations, it is sometimes useful to be able to identify the point that corresponds to a particular value of the parameter. In the next activity you will see how to use Mathcad to identify a point on an ellipse.

Activity 6.2 Identifying a particular point on an ellipse

You should still be working with Mathcad file 221A2-01.

Move to page 5 of the worksheet and carry out Task 3.

A solution and comments are provided on page 6 of the worksheet, and there are two further comments below.

Comment

- If you enter expressions involving an ordinary, single-valued variable (instead of a range variable) in the x- and y-placeholders of a graph, then the result is a plot of a single point. The point can be seen only after the graph trace has been formatted to display it as a symbol. This is the technique that was used to identify a particular point on the ellipse.
- As the value of T increases through the range $0 \leq T \leq 2\pi$, the point $(8\cos T, 5\sin T)$ travels once anticlockwise around the ellipse, starting and finishing at $(8, 0)$. In particular, $T = \pi/2$ corresponds to the point $(0, 5)$, $T = \pi$ to $(-8, 0)$, and $T = 3\pi/2$ to $(0, -5)$.

Now close Mathcad file 221A2-01.

6.2 Parabolas

See Chapter A2, Subsection 5.1.

The standard parametrisation of the parabola is

$$x = at^2, \quad y = 2at.$$

This is a parametrisation of the parabola in standard position with equation

$$y^2 = 4ax \quad (\text{where } a > 0).$$

In Activity 6.3 you are asked to use Mathcad to plot such a parabola, with $a = 1$. There is no prepared Mathcad file for this activity – you are asked to create your own Mathcad worksheet. As with every worksheet that you create, it is a good idea to include some text, to make it more comprehensible to a reader such as your tutor (or yourself at a later date!).

Activity 6.3 Plotting a parabola from parametric equations

The instructions below tell you how to create a Mathcad worksheet containing a plot of the parabola with parametrisation $x = t^2$, $y = 2t$.

See *A Guide to Mathcad* if you require more details on creating and editing your own worksheets.

Part (b) describes how to enter a title in the worksheet, and you should also enter any other text that you think is appropriate. For example, when you define a variable, it is helpful to include some text nearby which indicates what the variable represents.

(a) Begin by creating a new worksheet, as follows. Select the **File** menu and choose **New...**. In the list of templates that appears, **Normal** should be selected by default. If not, click on it. Then click on the **OK** button to create a new (Normal) worksheet. (Alternatively, type `[Ctrl]n`, or click on the **New** button on the standard toolbar.)

If you have just started Mathcad running, then there is no need to do this, as it automatically starts with a new (Normal) worksheet.

(b) Enter a title at the top of your worksheet. To do this, first select the **Insert** menu and then choose **Text Region**. (Alternatively, type a double-quote ", given by `[Shift]2`.) Then type a suitable title – for example, `Graph of parabola` in the text box. To finish, click anywhere outside the text box or press `[Ctrl][Shift][Enter]`. If you need to edit the text later, simply click on it.

Before you enter each new item in the worksheet, you should position the red cross cursor in an appropriate place. However, anything that you enter can be moved, or deleted, if you wish.

(c) Define a range variable t going from -5 to 5 in steps of 0.1. You can use the buttons on the 'Calculator' and 'Matrix' toolbars to do this, or just type `t:-5,-4.9;5`. Remember that the second number in the definition is the 'next value' in the range, not the step size.

(d) Create your graph, positioning it below the definition of the range variable. To plot the parabola, enter t^2 in the x-axis placeholder (for example, type `t^2`) and $2t$ in the y-axis placeholder (type `2*t`).

You can click on the button on the 'Graph' toolbar, or type `@`, or use the **Insert** menu, **Graph ▶ X-Y Plot**.

Fix the scales of your graph from 0 to 20 on the x-axis, and from -10 to 10 on the y-axis, and resize the graph appropriately.

(e) Select the **File** menu and use **Save As...** to name and save your worksheet. (You will be working on it again in Activity 6.4.)

Comment

A solution is shown below.

Graph of parabola

Graph range $t := -5, -4.9 .. 5$

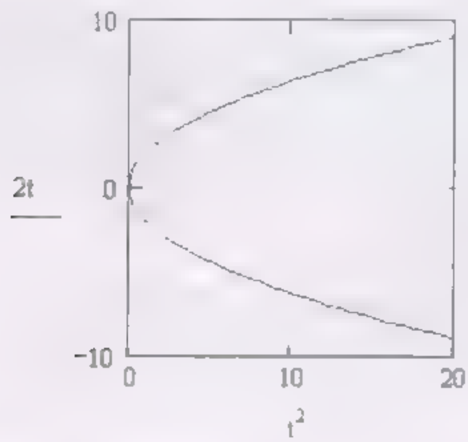

Figure 6.1 The parabola $y^2 = 4x$ in Mathcad

- After the graph scales have been fixed as described in part (d) of the activity, the graph is 20 units wide and 20 units high. It should therefore be resized to make the graph box square, as shown above.
- When the axis scales are fixed as described, there are a few points on the parabola that Mathcad calculates but does not plot. For example, for the final value in the graph range, $t = 5$, Mathcad calculates the point $(25, 10)$. This point lies beyond the right-hand edge of the graph, so it is not plotted. When you use Mathcad to plot a graph, it does not matter if some points lie outside the graph box in this way. However, it is preferable (if not always possible!) to choose a combination of graph range and axis scales which avoids the calculation of a large number of points that lie outside the graph box. You should also try to avoid the calculation of points that lie a long way outside the graph box.

In the next activity you are asked to identify a particular point on the parabola that you plotted in Activity 6.3.

Activity 6.4 Identifying a particular point on a parabola

You should still be working with the Mathcad file that you created in Activity 6.3.

Identify a particular point on the parabola $x = t^2$, $y = 2t$, by plotting a second trace consisting of a single point $(T^2, 2T)$, as follows.

(a) First define a variable T with value -4: make sure that this definition is positioned above your graph.

You may wish to make space for this new definition. To do so, position the red cross cursor where you want to insert some blank lines into the worksheet, then press the `[Enter]` key repeatedly until the space created is adequate for your needs. (Each key press will insert one blank line.)

(b) Position the vertical editing line at the right-hand end of the expression t^2 on the x-axis of your graph, with the horizontal editing line under the whole expression. (One way to do this is to click anywhere on the expression, then press `[Space]` and `[Insert]`, as necessary, to select it all.) Type a comma, then enter T^2 (for example, type **,T^2** - make sure that you type a capital T here).

In the same way, position the vertical editing line at the right-hand end of the expression $2t$ on the y-axis and type a comma, then enter $2T$ (type **,2*T**).

(c) Format the second trace to display the identified point as a box symbol. (Click in the graph to select it, and choose **Graph ▶ X-Y Plot...** from the **Format** menu (or just double-click in the middle of the graph) to bring up the 'Formatting Currently Selected X-Y Plot' option box. Then choose the 'Traces' tab; then click on 'trace 2' and set the 'Symbol' to **box** and 'Type' to **points**. Click on the **OK** button to finish.)

(d) Save your worksheet.

You may now like to increase the value of T and check that the point $(T^2, 2T)$ appears where you expect.

Comment

A solution is shown below.

Graph of parabola

Graph range $t := -5, -4.9 .. 5$

Particular value of the parameter t $T := -4$

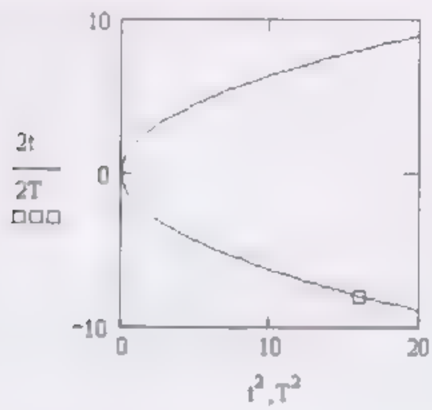

Figure 6.2 Identifying a particular point on $y^2 = 4x$

As T increases through the negative numbers towards 0, the point $(T^2, 2T)$ moves along the lower arm of the parabola towards the origin; when $T = 0$ it is at the origin; and as T increases through the positive numbers it moves away from the origin along the upper arm of the parabola.

If you found that you chose a value for T which caused the identified point to disappear from the graph, this was probably because the point lay outside the graph box! You can display its coordinates by evaluating T^2 and $2T$ in the worksheet. You can see more of the parabola, and see the identified point for more values of T, by changing the graph range and axis scales appropriately.

Now close the Mathcad file that you have created.

If you have made unsaved changes then you will see a dialogue box that asks whether you wish to save the changes.

6.3 Hyperbolas

The standard parametrisation of the hyperbola is

$$x = a \sec t, \quad y = b \tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

This is a parametrisation of the hyperbola in standard position with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{where } a, b > 0).$$

In the next activity you are asked to plot such a hyperbola, with $a = 1$ and $b = 2$. Plotting a hyperbola is a little more tricky than plotting an ellipse or a parabola, because you have to plot the two branches as separate traces.

You will use one range variable, tr, to plot the right-hand branch of the hyperbola, which corresponds to the range $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, and a second range variable, tl, to plot the left-hand branch of the hyperbola, which corresponds to the range $\frac{1}{2}\pi < t < \frac{3}{2}\pi$.

Activity 6.5 Plotting a hyperbola from parametric equations

Open Mathcad file **221A2-02 Hyperbolas in parametric form**, and carry out Task 1 on page 2 of the worksheet.

Comments are provided in the worksheet, and there are further comments below.

Comment

- The graph ranges $tr := -1.55, -1.54 \,..\, 1.55$ and $tl := 1.59, 1.60 \,..\, 4.69$ were chosen to ensure that no value of tr or tl is very close to $-\pi/2$, $\pi/2$ or $3\pi/2$. This avoids the calculation of points with very large coordinates.
- You may think that the initial graph shown in the comments in the Mathcad worksheet looks rather strange. This arises from the fact that if you do not enter numbers in the axis limit placeholders when you plot a graph, then by default Mathcad chooses limits which ensure that *all* the points that it calculates are plotted; it also chooses them to be appropriately round numbers. In this case, the endpoints of the portion of the hyperbola corresponding to the ranges of tr and tl are

$$(\sec(-1.55), 2\tan(-1.55)) \simeq (48.1, -96.2),$$
$$(\sec 1.55, 2\tan 1.55) \simeq (48.1, 96.2),$$
$$(\sec 1.59, 2\tan 1.59) \simeq (-52.1, -104.1),$$
$$(\sec 4.69, 2\tan 4.69) \simeq (-44.7, 89.3).$$

  When the axis limits are set closer to zero, as suggested in the worksheet, the part of the curve near the origin can be seen more clearly, and the shape appears much more familiar!

Mathcad notes

The two-letter names tr and tl were used for the range variables because the 'r' and 'l' suggest their use in plotting the right and left branches of the hyperbola. However, we could equally well have used single-letter names, such as t and u, or names consisting of a letter followed by a number, such as $t1$ and $t2$. Note that care is needed when you use the letter 'l' in Mathcad, as it looks very like the number '1'! In general, Mathcad variable names can be any combination of letters and numbers, but they must start with a letter.

In the next activity you are asked to add graph axes and asymptotes to the hyperbola that you plotted in Activity 6.5.

The equations of the axes are $x = 0$ and $y = 0$. These can be plotted using the parametric equations

$$x = 0,\ y = s \quad \text{and} \quad x = s,\ y = 0,$$

respectively.

The equations of the asymptotes of the hyperbola that you plotted in Activity 6.5 are $y = 2x$ and $y = -2x$. These can be plotted using the parametric equations

$$x = s,\ y = 2s \quad \text{and} \quad x = s,\ y = -2s,$$

respectively.

Activity 6.6 Adding graph axes and asymptotes

Move to page 3 of the worksheet and carry out Task 2.

Comments are provided in the worksheet.

You should still be working with Mathcad file 221A2-02.

Mathcad notes

◇ There are four ways to add axes to a Mathcad graph: you can plot extra lines, format the graph to turn 'Show Markers' on, format the graph to turn 'Grid Lines' on, or format the graph to use the 'Crossed' axis style. You used the first of these in this activity. Details of the other three methods can be found in *A Guide to Mathcad.*

◇ Note that the lines plotted for the axes may not align exactly with the tick marks for 0 at the edges of the graph box. The computer screen is made up of individual pixels (dots) which affect how accurately graphical information can be displayed, and there may be a tiny gap (of one or two pixels) between the lines and tick marks here.

In the next activity you will explore how the shape of the hyperbola changes as you vary a and b.

Activity 6.7 Exploring the shape of the hyperbola

Move to page 4 of the Mathcad worksheet. This page contains a plot of the hyperbola with parametrisation

You should still be working with Mathcad file 221A2-02.

$$x = a\sec t, \quad y = b\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi)$$

together with its asymptotes, and the graph axes. It is set up so that you can vary the values of a and b.

(a) Keep a fixed at the value 1, and try the values 1, 2 and 3 for b. Describe the effect on the hyperbola.

(b) Keep b fixed at the value 1, and try the values 1, 2 and 3 for a. Describe the effect on the hyperbola.

(c) Explain the effects that you saw in parts (a) and (b).

(d) Try the following pairs of values for a and b: $a = 1$, $b = 2$, and $a = 2$, $b = 4$. What is the relationship between the shapes of the two hyperbolas? You may find it helpful to print them out so that you can compare them.

The point with parameter $t = T$ is identified on the graph; you may like to try varying the value of T as well.

Solutions are given on page 34.

Comment

As T increases through the range $-\frac{1}{2}\pi < T < \frac{1}{2}\pi$, the point $(a\sec T, b\tan T)$ moves upwards along the right-hand branch of the hyperbola, lying on the x-axis when $T = 0$. Similarly, as T increases through the range $\frac{1}{2}\pi < T < \frac{3}{2}\pi$, the point $(a\sec T, b\tan T)$ moves upwards along the left-hand branch of the hyperbola, lying on the x-axis when $T = \pi$. Values of T close to $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ correspond to points that do not lie within the graph box.

Mathcad notes

- ◇ You may have noticed that both graph scales are fixed from $-S$ to S, where the variable S is defined at the top of page 4 of the Mathcad worksheet. This has the advantage that the graph scales can be changed just by changing the value of S - there is no need for any changes to the graph itself. Moreover, there is no need to change the range variable s which is used as the graph range to plot the axes and the asymptotes. This is because s is defined as $s := -S, -S+0.1 .. S$ (that is, s ranges from $-S$ to S in steps of size 0.1), so the values taken by s will automatically be updated if S changes, and the axes and asymptotes will always be drawn to the the edges of the graph box. This is a useful general technique.
- ◇ When a Mathcad graph is to include several curves, it is often best to plot them in the order 'least important first'. This is done for the graph that you used in this activity - the axes are plotted as traces 1 and 2, the asymptotes as traces 3 and 4, the branches of the hyperbola as traces 5 and 6, and the identified point as trace 7. The reason is that Mathcad draws trace 1 first, then trace 2 over the top of that, and so on. So a later trace may obscure some of an earlier trace. In fact, the trace order of the graph that you used in this activity does not have much effect, as the branches of the hyperbola cross the x-axis only once, and never cross the asymptotes!

Now close Mathcad file 221A2-02.

6.4 Focus, directrix and eccentricity (Optional)

The final, optional, activity in this section invites you to explore the shape of a conic with a fixed focus and directrix, and variable eccentricity. In particular, you can see the conic change between the three different types, ellipse, parabola and hyperbola, as the value of the eccentricity changes. The Mathcad worksheet also allows you to explore the focus–directrix property of the conics that it plots.

Activity 6.8 Focus, directrix and eccentricity (Optional)

Open Mathcad file **221A2-03 Focus directrix and eccentricity**. The worksheet shows a plot of the conic with focus $(1, 0)$, directrix $x = -1$, and eccentricity e. The eccentricity e is initially set to 0.8, but you can vary its value.

A point P on the conic is identified by a blue box symbol; you can change its position by changing the value of the variable T. Two line segments meet at P; the length of the horizontal segment is the distance Pd of P from the directrix d, and the length of the other segment is the distance PF of P from the focus F. The two distances PF and Pd, and the ratio PF/Pd, are evaluated at the bottom right of the worksheet.

(a) Vary the value of the eccentricity e, and observe how the shape of the conic changes. Some possible values for e are suggested in the worksheet, and you may like to try others of your own. In particular, you may like to explore the range of values of e for which it is difficult to distinguish by eye whether the conic is an ellipse, hyperbola or parabola.

You may wish to look at a 'small' conic in more detail, or to see more of a conic that extends outside the graph box. You can decrease or increase the part of the plane corresponding to the graph box by changing the value of the variable S.

You may notice that a small gap appears in some cases between the upper and lower halves of the conic. This is explained, along with other aspects of the worksheet, at the end of this subsection.

(b) Vary the value of T (keeping e fixed) and observe how, although the distances PF and Pd vary as the point P moves along the conic, their ratio PF/Pd remains constant — equal to the eccentricity e.

Note that not every value of T corresponds to a point on the conic. In fact, the x-coordinate of the identified point is equal to T, so by looking at the graph scales you can see roughly which values of T correspond to points on the conic. If you choose a value of T that does not correspond to a point on the conic, then no point is plotted.

For example, with $e = 0.8$, you can see that values of T between about 0 and 9 correspond to points on the conic (the exact range is $\frac{1}{9} \leq T \leq 9$).

Note that the graph is set up to identify points P on the upper half of the conic only.

Comment

When e is not much greater than 0, the conic is an ellipse, nearly circular in shape. It is positioned to the right of the y-axis, with two vertices on the x-axis. As e increases through the range $0 < e < 1$, both of these vertices move along the x-axis: the left one moves further left and approaches the origin, while the right one moves further right, with the result that the ellipse becomes increasingly elongated. When $e = 1$, the vertex that was approaching the origin has reached it, while the other vertex has 'disappeared to infinity'. The conic is now a parabola in standard position. As e increases through the range $e > 1$, the vertex near the origin continues to move left along the x-axis, away from the origin, while the other vertex has 'reappeared' on the negative part of the x-axis, and moves right, approaching the first vertex. The conic is a hyperbola, whose asymptotes become increasingly steep.

If you are interested in how the graph in the worksheet associated with Activity 6.8 is achieved, then you may like to read the following explanation. Otherwise, just close Mathcad file 221A2-03.

Explanation of the Mathcad worksheet

The worksheet states that the conic with focus $(1, 0)$, directrix $x = -1$ and eccentricity e is represented by the equation

$$\alpha x^2 - y^2 + 2\beta x + \alpha = 0, \tag{6.1}$$

where $\alpha = e^2 - 1$ and $\beta = e^2 + 1$.

This equation can be obtained using the method of the solution to Chapter A2, Activity 3.5.

If we then set $x = t$ and solve equation (6.1) for y, we obtain $y = \pm\sqrt{\alpha(t^2 + 1) + 2\beta t}$. Thus we can plot the conic on a Mathcad graph using two separate traces: the first with x-coordinate t and y-coordinate $\sqrt{\alpha(t^2 + 1) + 2\beta t}$, and the second with x-coordinate t and y-coordinate $-\sqrt{\alpha(t^2 + 1) + 2\beta t}$. These are traces 5 and 6 in the Mathcad worksheet. Now t is defined to range from $-S$ to S (where S is defined earlier in the worksheet), but not all of these values of t correspond to points on the conic.

So what happens for the other values of t? In fact, t corresponds to a point on the conic if and only if $\alpha(t^2+1)+2\beta t$ is non-negative. If $\alpha(t^2+1)+2\beta t$ is negative, then according to its definition, the y-coordinate of the point to be plotted is the square root of a negative number. This does not make sense, and so Mathcad does not plot a corresponding point on the graph.

The coordinates of the identified point P on the graph are $x = T$, $y = \sqrt{\alpha(T^2+1)+2\beta T}$, so the identified point always lies on trace 5, the upper trace. If T is set to a value that does not correspond to a point on the conic, then no point is plotted.

We end with explanations of three other aspects of the graph.

- The line segments indicating the distances PF and Pd on the graph are drawn by plotting the three points $(u_0, v_0) = (1, 0)$, $(u_1, v_1) = (x, y)$ and $(u_2, v_2) = (-1, y)$, with a line trace to join them together. This is trace 7 on the graph. The calculations are off the Mathcad page to the right.

Note that
$$\frac{S-(-S)}{0.002S} = 1000.$$

- The graph range is defined as $t := -S, -0.998S \,..\, S$; that is, t ranges from $-S$ to S in steps of $0.002S$. This gives a way of plotting 1000 points, whatever the value chosen for S.

- In some cases a point where the conic intersects the x-axis does not correspond with a value of t in the chosen graph range. Then Mathcad cannot plot the corresponding point on the x-axis, which may lead to a slight gap between the upper and lower traces for the conic close to that point.

  For example, with $e = 2.7$, the hyperbola cuts the x-axis at $(-\frac{17}{37}, 0)$, but $-\frac{17}{37} = -0.459\ldots$ is not a value taken by t within the graph range. The closest such values are $t = -0.46$, for which no point is plotted, and $t = -0.44$, for which the corresponding points on the conic are approximately $(-0.44, \pm 0.46)$. Hence a gap of height $2 \times 0.46 = 0.92$ appears on the Mathcad graph.

This assumes that $S = 10$ in the worksheet, so that t takes the values
$$-10, -9.98, \ldots, 10.$$

Now close Mathcad file 221A2-03.

Chapter A3, Section 5
Isometries on the computer

In this section, you will see how Mathcad can demonstrate the effect of applying isometries in $\mathbb{R}^2$. First isometries are applied to triangles. Then Mathcad is used to plot the graphs of conics with xy-terms, by applying isometries to simpler conics.

There is also an *optional* subsection in which you are invited to use Mathcad to explore surface and contour plots of functions of two variables.

5.1 Isometries and triangles

In this subsection you will use Mathcad to explore the effect of isometries on triangles. The notation used in the Mathcad worksheet for this subsection differs in some respects from that used in the earlier sections of Chapter A3. These changes are needed to allow isometries to be implemented and composed effectively in Mathcad.

The main difference is that points in $\mathbb{R}^2$ are represented using **vector notation**: for our current purposes, a **vector** is a column of numbers. For example, to define the point P with coordinates $(5, -2)$ as a vector in Mathcad, we use

A vector is a particular type of **matrix** (a rectangular array of numbers). You will study matrices in detail in Block B.

$$P := \begin{pmatrix} 5 \\ -2 \end{pmatrix} \tag{5.1}$$

You can think of P as representing a sequence of two numbers $P_0 = 5$ and $P_1 = -2$, written in a vertical table. Thus P_0 is the x-coordinate of P and P_1 is the y-coordinate of P.

Remember that in Mathcad the first term of a sequence has subscript 0, by default.

We define an isometry in Mathcad as a function whose inputs and outputs are vectors. For example, to define the translation t that moves each point two units to the right and one unit down, we use

In Chapter A3, Section 2, we used $t_{2,-1}$ for this translation, but in Mathcad this notation is not convenient.

$$t(P) := \begin{pmatrix} P_0 + 2 \\ P_1 - 1 \end{pmatrix} \tag{5.2}$$

Once this definition has been made in a Mathcad worksheet, the function t can be applied to any input vector. For example, if P and t have been defined as in equations (5.1) and (5.2), then evaluating $t(P)$ gives $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$. Also, evaluating $t(P)_0$ gives 7 and evaluating $t(P)_1$ gives -3.

Vector notation is used throughout the Mathcad worksheet for this subsection, but you will not need to edit any vectors in the worksheet, nor to create any new ones.

You will learn how to enter and edit vectors, and matrices in general, in Block B. Instructions are also given in *A Guide to Mathcad.*

In the first activity you will see vector notation used to implement a translation in Mathcad and demonstrate its effect on a triangle.

Activity 5.1 Points, triangles and the translation function

Open Mathcad file **221A3-01 Isometries and triangles**, and move to page 2 of the worksheet. Read this page carefully – it introduces the notation that will be used throughout the worksheet.

To check that you understand the way in which the translation function has been implemented, try changing the values of c and d so that the image triangle is in a different position, for example, with its left-most vertex at the origin.

Comment

- ◇ The purpose of the 'point function'

  $$P(n) := \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

  defined in the worksheet, is to give an easy way to deal with the vertices (x_0, y_0), (x_1, y_1), (x_2, y_2) and $(x_3, y_3) = (x_0, y_0)$ of the triangle. It means that $P(n)$ is the nth vertex, and $t(P(n))$ is the image of the nth vertex under the translation t. The x-coordinate of the nth vertex is then $P(n)_0$, and its y-coordinate is $P(n)_1$. Similarly, the x-coordinate of the image of the nth vertex under t is $t(P(n))_0$, and its y-coordinate is $t(P(n))_1$. You can see these expressions used on the axes of the graph to plot the two triangles.
- ◇ In the Mathcad graph, the original triangle is plotted as a solid black trace and the image triangle as solid red. If you have difficulty distinguishing between the triangles or wish to print the graph on a non-colour printer, then you may prefer to change the line style of the image triangle trace from 'solid' to 'dot' or 'dash'.

Similar advice applies to many of the Mathcad graphs that you will see in this section. See the solution to Activity 5.5, for example.

Mathcad notes

The graph at the bottom of the Mathcad page has been resized and both scales have been fixed from 0 to 10. It has also been formatted to show grid lines - this was done by switching on 'Grid Lines', then switching off 'Auto Grid' and setting the 'Number of Grids' to 10 for each axis.

The Mathcad notation that you saw in Activity 5.1 is used throughout the rest of the Mathcad worksheet to implement isometries and demonstrate their effects on triangles.

Remember that when plotting the effects of isometries it is important to have equal scales on both axes.

In the next activity you will see the effect of a rotation demonstrated in Mathcad. The rotation is implemented by defining an appropriate value for the angle θ and setting

$$r(P) := \begin{pmatrix} P_0 \cos(\theta) - P_1 \sin(\theta) \\ P_0 \sin(\theta) + P_1 \cos(\theta) \end{pmatrix}.$$

Remember that P_0 and P_1 are the x- and y-coordinates of P.

Activity 5.2 Rotations

Move to page 3 of the Mathcad worksheet. A triangle is defined near the top of the page. Further down, the rotation $r = r_\theta$ is defined, and a graph shows the effect of r_θ on the triangle when $\theta = \frac{1}{2}\pi$. You can change the value of θ, and you can enter this angle in either radians or degrees, as explained in the worksheet.

You should still be working with Mathcad file 221A3-01.

(a) Figure 5.1 shows the original triangle defined in the worksheet, and its images under three rotations, namely $r_{3\pi/4}$, $r_{-\pi/4}$ and $r_{-5\pi/6}$. Try to decide which image corresponds to which rotation. Then confirm your answer by trying each of the three values of θ in turn in the worksheet and checking that the image appears where you expect.

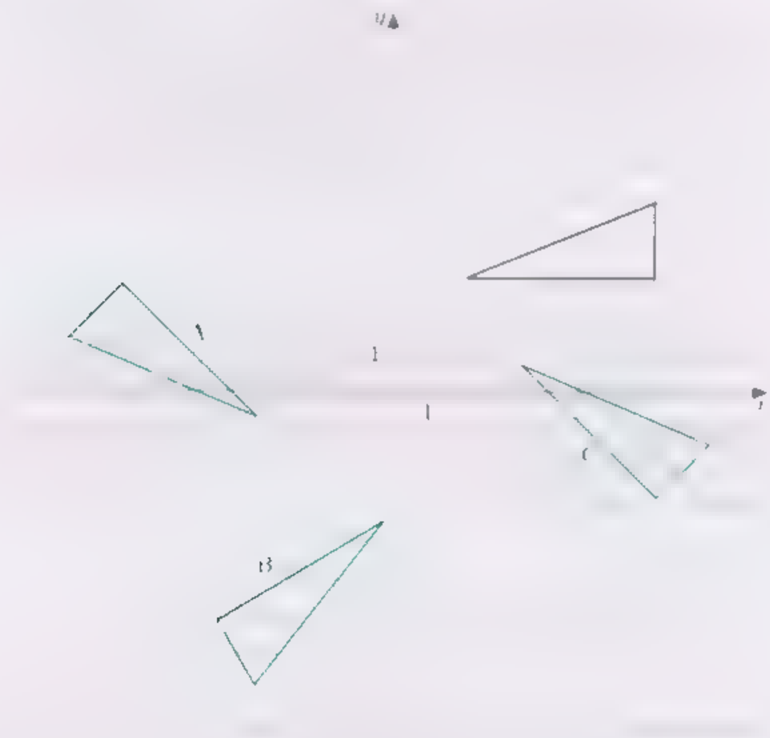

Figure 5.1

(b) Repeat part (a) for Figure 5.2, which shows the images of the triangle under rotations through 60°, −160° and −100°.

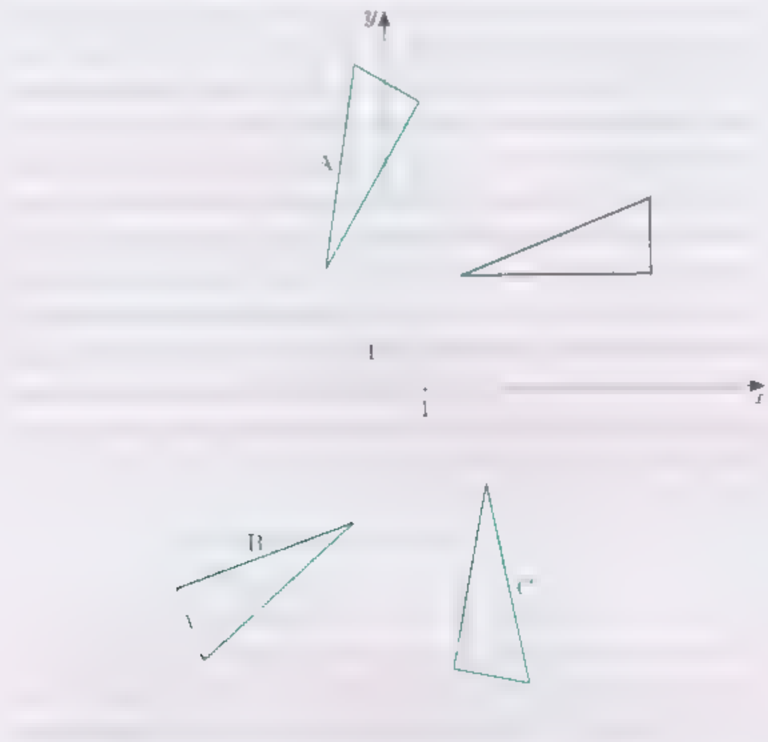

Figure 5.2

Solutions are given on page 34.

Comment

Note that traces 1 and 2 on the graph plot the x- and y-axes, using the parametrisations $x = s$, $y = 0$ and $x = 0$, $y = s$.

Mathcad notes

- Note that deg (in lower-case letters) is a built-in Mathcad constant. Its value is 0.017 45 . . . , the number of radians in one degree. For example, evaluating 180 deg gives 3.141 59 . . . . (The constant deg is included in the system of units built into Mathcad, alongside standard SI units, such as m for metre and kg for kilogram. So deg is *not* defined in worksheets where these units are switched off – **Math** menu, **Options...**, 'Unit System' tab, Default Units 'None'.)
- You may find it helpful to reformat the graph to add grid lines. To do this, switch 'Grid Lines' on, switch 'Auto Grid' off, and set 'Number of Grids' to 20 for both axes. (You can also choose the 'Grid Color...' – the default colour for the grid lines is green.) However, displaying this number of grid lines makes the axis labelling rather overcrowded, and may make the graph less clear if printed.

In the next activity you will see the effect of a reflection demonstrated in Mathcad.

Activity 5.3 Reflections

Move to page 4 of the Mathcad worksheet. This page is set out in a similar way to page 3. The reflection q_ϕ is defined, and a graph shows the effect of q_ϕ on the specified triangle, together with the line of reflection. You can change the value of ϕ.

You should still be working with Mathcad file 221A3-01.

We use θ for the angle of a rotation and ϕ for the angle of a reflection, to avoid confusion.

(a) Figure 5.3 shows the original triangle defined in the worksheet, and its images under three reflections, namely $q_{5\pi/6}$, $q_{3\pi/4}$ and $q_{3\pi/8}$. Try to decide which image corresponds to which reflection. Then confirm your answer by trying each of the three values of ϕ in turn in the worksheet and checking that the image appears where you expect.

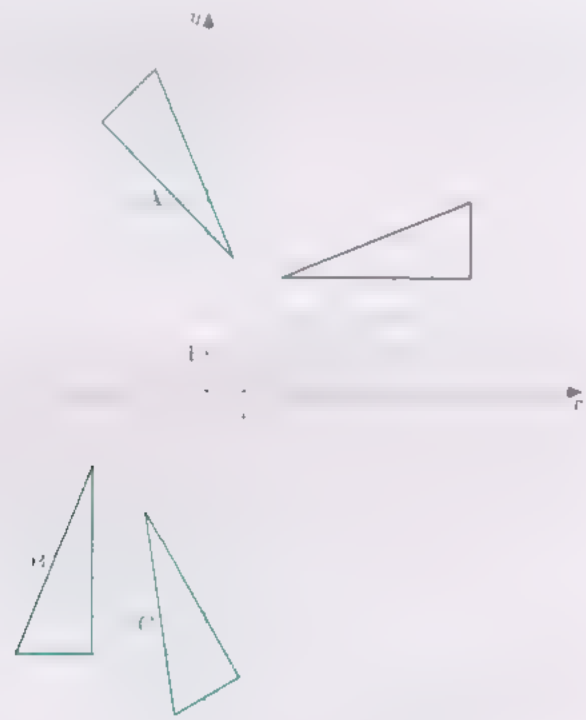

Figure 5.3

(b) Edit the Mathcad page to change the original triangle to the triangle with vertices (0.9, 6.2), (5.7, 2.6) and (6.3, 8.1), and set ϕ to 45°. Then, by experimenting with different values of ϕ, find to the nearest degree the angle ϕ such that the corresponding reflection maps the triangle onto itself.

Recall that you can enter an angle in degrees by using the built-in Mathcad constant 'deg'. For example, for an angle of 180°, enter 180 deg (type `180*deg`).

Solutions are given on page 34.

Comment

The line of reflection is plotted using the parametrisation

$$x = s, \quad y = s\tan\phi.$$

The line making the angle ϕ with the positive x-axis has slope $\tan\phi$.

In the next activity you will see the effect of composites of isometries demonstrated in Mathcad.

Activity 5.4 Composite isometries

You should still be working with Mathcad file 221A3-01.

Move to page 5 of the Mathcad worksheet. This page demonstrates the effect of the composite $g \circ f$, of an isometry f followed by an isometry g, on a triangle. Initially, f is defined to be the reflection q_ϕ where $\phi = 0$, and g to be the translation $t_{c,d}$ where $c = 1$ and $d = 0$, so $g \circ f$ is a glide-reflection parallel to the x-axis. You can see different glide-reflections by changing the values of ϕ, c and d appropriately (where ϕ is the angle that the line $y = (d/c)x$ makes with the positive x-axis).

Figure 5.4 shows the original triangle defined in the worksheet, and its images under three glide-reflections, namely $t_{0,2} \circ q_{\pi/2}$, $t_{-6,-6} \circ q_{\pi/4}$ and $t_{-11,0} \circ q_0$. Try to decide which image corresponds to which glide-reflection. Then confirm your answer by trying each of the three sets of values of ϕ, c and d in turn in the worksheet and checking that the final image appears where you expect.

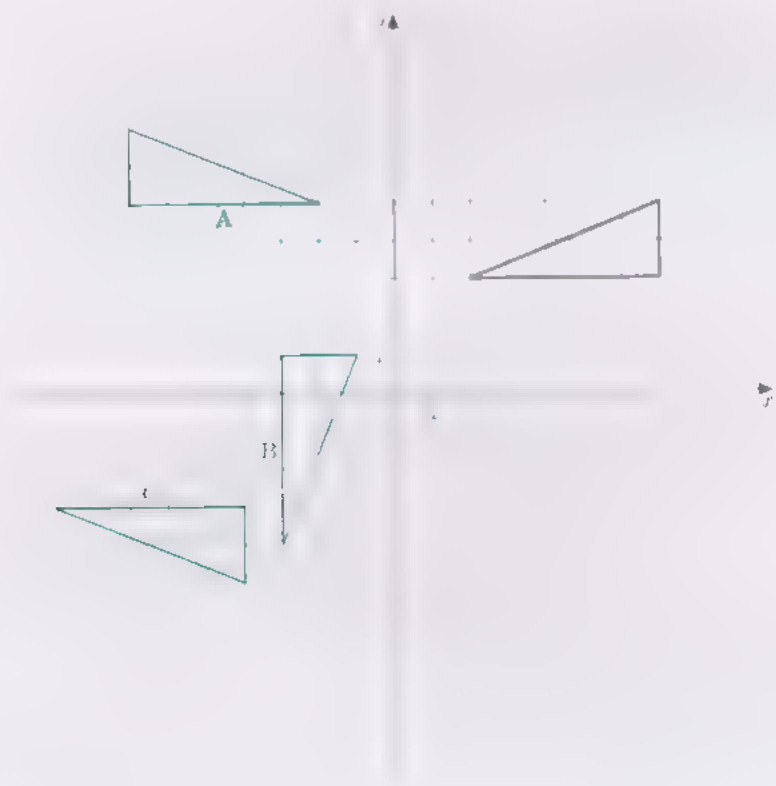

Figure 5.4

Solutions are given on page 34.

Comment

If you wish, you can experiment with the composites of different types of isometries, by changing the definitions of f and g.

Now close Mathcad file 221A3-01.

5.2 Isometries and conics

In the main text, we developed a method of sketching a quadratic curve L with an xy-term. This method involves first determining a conic K with no xy-term that is congruent to L, and a rotation that maps K onto L, then drawing a graph of K, and finally applying the rotation to obtain a graph of L. The final two stages of this process can be carried out using Mathcad, as you will see in this subsection.

See Chapter A3, Subsection 4.2.

The Mathcad worksheet for this subsection uses vector notation in a similar way to the worksheet for Subsection 5.1.

In the first activity you are asked to plot a quadratic curve that is the image under a rotation of an ellipse in reflected standard position. You will use a page of a Mathcad worksheet that has been set up to plot the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a, b > 0), \tag{5.3}$$

This ellipse is in standard position if $a > b$, and in reflected standard position if $b > a$.

and its image under a rotation r_θ, for values of a, b and θ that are defined in the worksheet and that you can change.

Activity 5.5 A rotated ellipse

Open Mathcad file **221A3-02 Isometries and conics**, read the introduction on page 1 of the worksheet, and move to page 2.

On this page, the ellipse with equation (5.3), and three isometries $t = t_{c,d}$, $r = r_\theta$ and $q = q_\phi$, are defined. An isometry f is initially set to be the rotation r, and the original ellipse and its image under f are plotted on a graph.

You can change the values of a and b to specify different ellipses. You can also change the definition of f to see the image of the ellipse under different types of isometries, and you can change the values of c, d, θ and ϕ to change the isometries themselves.

The quadratic curve L with equation

See Chapter A3, Example 4.1.

$$19x^2 + 6xy + 11y^2 - 40 = 0$$

is the image of the ellipse, in reflected standard position, with equation

$$\tfrac{1}{2}x^2 + \tfrac{1}{4}y^2 = 1$$

under a rotation of approximately 18° about the origin. Change the values of a, b and θ on page 2 of the worksheet, to obtain the graph of L.

Remember that rotations through a positive angle are anticlockwise.

A solution is given on page 34.

Comment

The original ellipse is plotted using the technique that you saw in the computer work for Chapter A2: a large number of points on it are calculated using a suitable range variable u and the usual parametrisation of the ellipse (which remains valid for an ellipse in reflected standard position). However, here we use vector notation to make it easy to deal

In Chapter A2 the name t was used for the range variable, but here we use u.

with the points on the ellipse and their images under the isometry f. Just as the 'point function' P defined the vertices of the triangle in Subsection 5.1, so here we have a 'parametrisation function' p, given by

$$p(u) := \begin{pmatrix} a\cos(u) \\ b\sin(u) \end{pmatrix},$$

which defines points on the ellipse. Each value of u corresponds to a point $p(u)$ on the ellipse, and the image of this point under the isometry f is $f(p(u))$. You can see these expressions used on the graph axes to plot the ellipse and its image under f; as usual, the subscripts 0 and 1 specify x- and y-coordinates, respectively.

Mathcad notes

- The square root operator can be obtained by clicking on the appropriate button on the 'Calculator' toolbar, or by typing \ (a backslash). For example, to obtain $\sqrt{2}$ you can type \2 .
- You cannot use the same name for a variable and a function in a Mathcad worksheet. Here we use t for a translation, so a different name is required for the range variable — we use u.

In the next activity you will use Mathcad to produce a graph of a hyperbola whose equation has an xy-term.

Activity 5.6 A rotated hyperbola

You should still be working with Mathcad file 221A3-02.

Move to page 3 of the Mathcad worksheet. It is set up in the same way as page 2, but the curve plotted is a hyperbola.

You could work this out for yourself by following the strategy given in Chapter A3, Subsection 4.2.

The quadratic curve L with equation

$$24x^2 + 20xy - 24y^2 - 13 = 0$$

is the image of the hyperbola with equation

$$2x^2 - 2y^2 = 1$$

under a rotation of approximately 11° about the origin. Change the values of a, b and θ on page 3 of the worksheet, to obtain the graph of L.

A solution is given on page 34.

We now discuss quadratic curves that are the image under a rotation of a hyperbola in *reflected* standard position.

A hyperbola in reflected standard position does not have parametric equations of the the same form as those for a hyperbola in standard position, so the original hyperbola on the Mathcad page will never be in reflected standard position.

Instead, we can start with a hyperbola in standard position, reflect it in the line $y = x$ to obtain a hyperbola in reflected standard position, and then rotate it through an appropriate angle to obtain a graph of the required quadratic curve. You are asked to do this in the next activity.

Activity 5.7 A reflected and rotated hyperbola

The quadratic curve L with equation

$$x^2 + 12xy + 6y^2 - 30 = 0$$

is the image of the hyperbola K in reflected standard position with equation

$$-\frac{x^2}{10} + \frac{y^2}{3} = 1,$$

under a rotation of approximately $-34°$ about the origin. This hyperbola is in turn the image of the hyperbola H in standard position with equation

$$\frac{x^2}{3} - \frac{y^2}{10} = 1$$

under reflection in the line $y = x$, that is, under $q_{\pi/4}$.

(a) On page 3 of the Mathcad worksheet, set $\theta = 0$ (if it is not already set to 0), and change the values of a and b to obtain a graph of H. The original and image hyperbolas will then be superimposed.

(b) In the definition of the isometry f, change r to q; this means that f is the isometry q_ϕ. Then change the value of ϕ to obtain a graph of the hyperbola K.

(c) Now edit the definition of the isometry f again so it reads as follows:

$$f(P) := r(q(P)).$$

This means that f is the isometry $r_\theta \circ q_\phi$. Change the value of θ to obtain a graph of L.

Solutions are given on page 35.

See Chapter A3, Activity 4.3.

Reflecting in the line $y = x$ corresponds to exchanging the roles of the x- and y-axes.

You should still be working with Mathcad file 221A3-02.

Note that in order to obtain the graph of L, you only need to set the values of a, b, ϕ and θ, and edit the definition of f as in part (c).

Comment

Since reflecting a hyperbola in standard position in the line $y = x$ produces the same image as rotating it about the origin through the angle 90°, an alternative way to obtain a graph of the above quadratic curve L is to begin with the same hyperbola H in standard position, and simply rotate it through $90° + (-34)° = 56°$.

In the next activity you will use Mathcad to produce a graph of a parabola with an xy-term. Here, we need to use a translation as well as a rotation.

Activity 5.8 A translated and rotated parabola

Move to page 4 of the Mathcad worksheet. It is set up in the same way as pages 2 and 3, but the curve plotted is a parabola and the isometry f is set initially as a translation.

The quadratic curve L with equation

$$9x^2 - 24xy + 16y^2 - 10x - 70y - 75 = 0,$$

is the image of the quadratic curve K with equation

$$y^2 - 2x - 2y - 3 = 0,$$

which has no xy-term, under a rotation of approximately 37° about the origin. Completing the square in the equation for K, and simplifying, gives

$$(y - 1)^2 - 2(x + 2) = 0,$$

You should still be working with Mathcad file 221A3-02

You could work this out for yourself by following the method explained in Chapter A3, Subsection 4.2. Note that in this case $D \neq 0$ and $E \neq 0$.

so the curve K is itself the image of the parabola H in standard position with equation

$$y^2 = 2x$$

under the translation $t_{-2,1}$.

(a) On page 4 of the worksheet, set $c = 0$ and $d = 0$ (if they are not already set to 0) and change the value of a to obtain a graph of H.

(b) Change the values of c and d to obtain a graph of K.

(c) Now edit the definition of the isometry f so that it reads as follows:

$$f(P) := r(t(P)).$$

This means that f is the isometry $r_\theta \circ t_{c,d}$. Change the value of θ to obtain the graph of L.

Solutions are given on page 35.

Now close Mathcad file 221A3-02.

5.3 Surface and contour plots (Optional)

In this subsection you are invited to use Mathcad to explore surface and contour plots of functions of two variables. A contour plot can provide a useful check on the shape and position of a quadratic curve.

See Chapter A3, Section 1.3.

Activity 5.9 Surface and contour plots (Optional)

Open Mathcad file **221A3-03 Surface and contour plots**, and read through pages 1 and 2 of the worksheet.

(a) In the definition of the function f on page 2 of the worksheet, change 'd' to 'g' so that the function g is plotted instead of the distance function d.

The quadratic curve with equation

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0$$

is then displayed as the contour with height 0.

This is the ellipse E discussed in Chapter A3, Section 1.

(b) Now use page 2 of the worksheet to verify the shapes of the quadratic curves that you considered in Activities 5.5 to 5.8, as follows.

For each quadratic curve, change the values of the coefficients in the worksheet to the values of A, B, C, D, E and F corresponding to the equation of the curve. Check that the shape and position of the contour with height 0 are the same as those of the curve.

(c) If you would like to see how to create and to format surface and contour plots, then read page 3 of the worksheet.

Now close Mathcad file 221A3-03.

Solutions to Activities

Chapter A1

Solution 4.3

The value of the expression F_{n+1}/F_n appears to tend to the golden ratio ϕ as n increases, and appears to lie alternately above and below ϕ.

(You met this property in Section 2 and will see a proof of it in Section 5.)

The value of the expression $F_n^2 + F_{n+1}^2$ always seems to be a Fibonacci number. These numbers appear in alternate positions in the F_n table. To be precise,

$$\begin{aligned} F_1^2 + F_2^2 &= F_3, \\ F_2^2 + F_3^2 &= F_5, \\ F_3^2 + F_4^2 &= F_7, \\ &\vdots \\ F_9^2 + F_{10}^2 &= F_{19}. \end{aligned}$$

This suggests the conjecture that, for $n = 1, 2, 3, \ldots$,

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

(This conjecture is true. It can be proved using Binet's formula, but we shall not do so.)

Solution 4.4

It appears that the sequence u_{n+1}/u_n always tends to one of the roots of the auxiliary equation, provided that these roots are real. In fact, if the roots are real and distinct, then the sequence seems to tend to the root of larger magnitude, whether this is positive or negative. (The *magnitude* of a real number x is x if $x \geq 0$ and $-x$ if $x < 0$. For example, the magnitudes of 3 and -3 are both 3.) It is natural to conjecture that this happens for any linear second-order recurrence sequence whose auxiliary equation has real roots. Changing the initial values a and b does not seem to affect this long-term behaviour.

(This conjecture is indeed true for all such linear second-order recurrence sequences u_n, but we prove it only for the particular case of the Fibonacci sequence, in Section 5.)

You may also have noticed that although for the Fibonacci sequence the terms of the sequence u_{n+1}/u_n appear to alternate above and below the value to which they tend, this does not happen for every linear second-order recurrence sequence.

(This behaviour is related to whether the roots of the auxiliary equation are of the same sign or of opposite signs.)

Solution 4.5

It appears that if $p = 1$, $q = 1$ and $a = 0$, then the sequence $u_n^2 + u_{n+1}^2$ $(n = 1, 2, 3, \ldots)$ consists of b times every other term of the original sequence u_n, starting with u_3. This suggests the conjecture that if $p = 1$, $q = 1$ and $a = 0$, then, for all b,

$$u_n^2 + u_{n+1}^2 = bu_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{1}$$

Similarly, it appears that if $p = 1$, $q = 1$ and $b = 0$, then the sequence $u_n^2 + u_{n+1}^2$ $(n = 1, 2, 3, \ldots)$ consists of a times every other term of the original sequence, starting with u_2. This suggests the conjecture that if $p = 1$, $q = 1$ and $b = 0$, then, for all a,

$$u_n^2 + u_{n+1}^2 = au_{2n}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{2}$$

If you tried varying p, then you would have found that this seems to have no effect on the above conjectures; that is, the first conjecture seems to be true if $q = 1$, $a = 0$ and for all p and b, and the second seems to be true if $q = 1$, $b = 0$ and for all p and a.

(The identities in equations (1) and (2) can in fact be deduced from the identity

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots,$$

which you met in Activity 4.3. For equation (1), we use the fact that if $p = 1$, $q = 1$ and $a = 0$, then

$$u_n = bF_n, \quad \text{for } n = 1, 2, 3, \ldots.$$

Therefore, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} u_n^2 + u_{n+1}^2 &= (bF_n)^2 + (bF_{n+1})^2 \\ &= b^2(F_n^2 + F_{n+1}^2) \\ &= b^2 F_{2n+1} \\ &= bu_{2n+1}. \end{aligned}$$

Equation (2) can be proved in a similar way, using the fact that if $p = 1$, $q = 1$ and $b = 0$, then

$$u_n = aF_{n-1}, \quad \text{for } n = 1, 2, 3, \ldots.$$

In fact, there is a general identity

$$qu_n^2 + u_{n+1}^2 = aqu_{2n} + bu_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots,$$

which holds for all values of p, q, a and b. This can be proved using the closed form of the sequence, but the details are a little involved.)

Chapter A2

Solution 6.7

(a) If a is fixed, then as b increases, the two points where the hyperbola crosses the x-axis remain fixed, and the asymptotes become steeper.

(b) If b is fixed, then as a increases, the two points where the hyperbola crosses the x-axis move away from the origin, and the asymptotes become less steep.

(c) The points where the hyperbola crosses the x-axis are $(\pm a, 0)$, and the asymptotes are $y = \pm(b/a)x$, where $a, b > 0$ (see Section 2). Thus if a is fixed, then the two crossing points remain fixed; if b now increases, then b/a increases and hence the asymptotes become steeper. On the other hand, if b is fixed and a increases, then the crossing points move away from the origin; also b/a decreases and hence the asymptotes become less steep.

(d) The two hyperbolas have the same asymptotes, since they have the same value of b/a. In fact, their shapes are even more closely related: they are similar. The first hyperbola has parametrisation

$$x = \sec t, \quad y = 2\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi)$$

and the second has parametrisation

$$x = 2\sec t, \quad y = 4\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi),$$

so if the point (u, v) lies on the first hyperbola, then the point $(2u, 2v)$ lies on the second.

(In general, any two hyperbolas with the same value of b/a are similar: each is obtained from the other by a scaling with the same scale factor with respect to both axes. This is also true for ellipses.)

Chapter A3

Solution 5.2

(a) A is the image of the triangle under $r_{3\pi/4}$, B is its image under $r_{-5\pi/6}$, and C is its image under $r_{-\pi/4}$.

(b) A is the image of the triangle under a rotation through 60°, B is its image under a rotation through −160°, and C is its image under a rotation through −100°.

Solution 5.3

(a) A is the image of the triangle under $q_{3\pi/8}$, B is its image under $q_{3\pi/4}$, and C is its image under $q_{5\pi/6}$.

(b) Reflection in the line through the origin that makes an angle of approximately 53° with the positive x-axis maps the triangle onto itself.

Solution 5.4

A is the image of the triangle under $t_{0,2} \circ q_{\pi/2}$, B is its image under $t_{-6,-6} \circ q_{\pi/4}$, and C is its image under $t_{-11,0} \circ q_0$.

Solution 5.5

Setting $a = \sqrt{2}$, $b = 2$ and $\theta = 18°$ gives the graph in Figure S3.1, in which L is the dashed trace. Here the axes limits have been changed to make the plot of the ellipses larger.

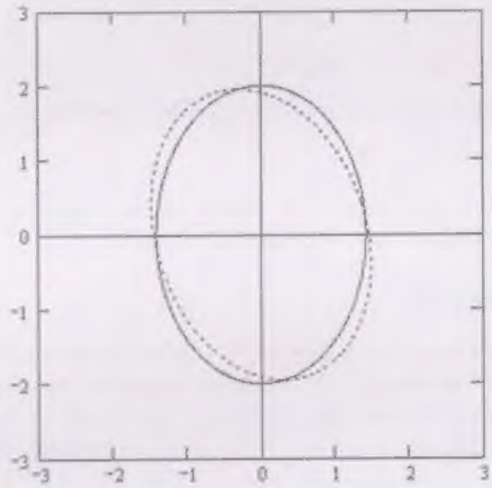

Figure S3.1

Solution 5.6

Setting $a = 1/\sqrt{2}$, $b = 1/\sqrt{2}$ and $\theta = 11°$ gives the graph in Figure S3.2, in which L is the dashed trace.

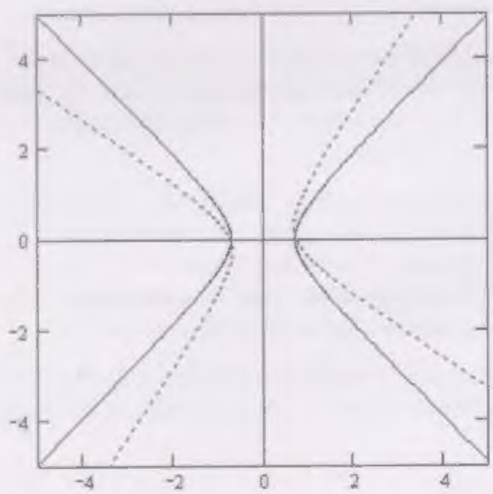

Figure S3.2

Solution 5.7

(a) Setting $\theta = 0$, $a = \sqrt{3}$ and $b = \sqrt{10}$ gives the graph of H, as in Figure S3.3.

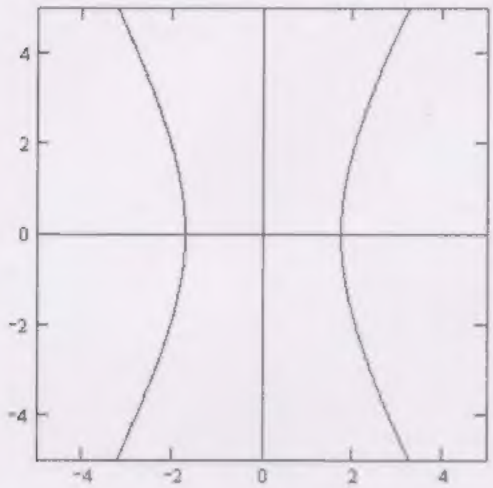

Figure S3.3

(b) Setting $\phi = 45°$ gives the graph in Figure S3.4, in which K is the dashed trace.

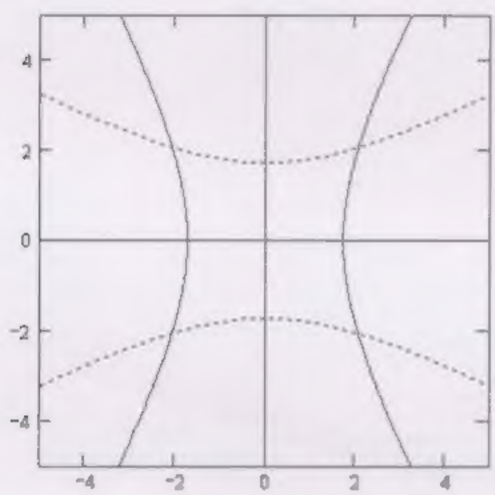

Figure S3.4

(c) Setting $\theta = -34°$ gives the graph in Figure S3.5, in which L is the dashed trace.

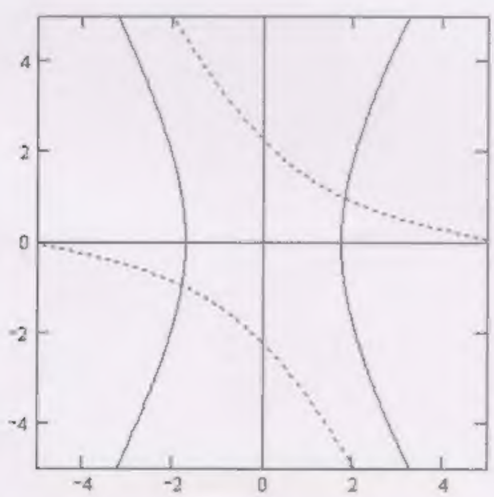

Figure S3.5

Solution 5.8

(a) Setting $c = 0$, $d = 0$ and $a = \frac{1}{2}$ gives the graph of H, as in Figure S3.6.

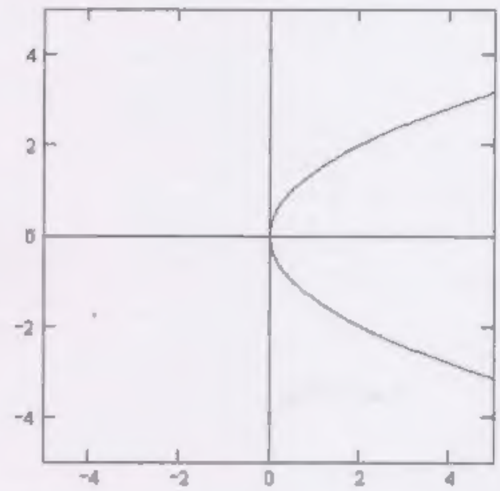

Figure S3.6

(b) Setting $c = -2$ and $d = 1$ gives the graph in Figure S3.7, in which K is the dashed trace.

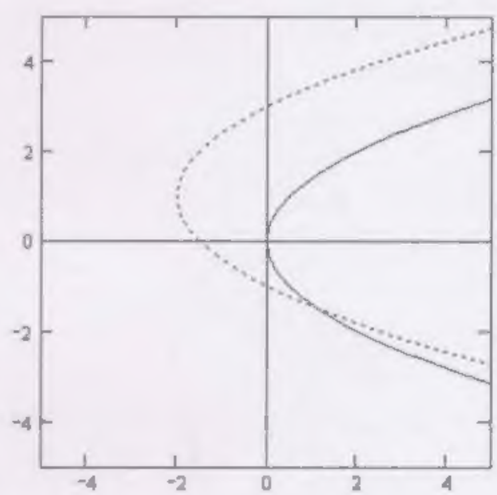

Figure S3.7

(c) Setting $\theta = 37°$ gives the graph in Figure S3.8, in which L is the dashed trace.

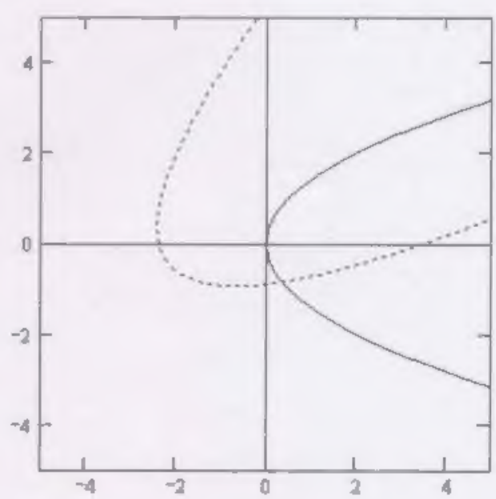

Figure S3.8

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ISBN 0 7492 6651 1